arXiv:hep-th/9512027v1 5 Dec 1995

PAR–LPTHE 95–56

B-V QUANTIZATION IN 2-D GRAVITY AND NEW CONFORMAL FIELDS

Laurent Baulieu *

LPTHE

Université Pierre et Marie Curie - PARIS VI

Université Denis Diderot - Paris VII

Laboratoire associé No. 280 au CNRS †

and

RIMS, Kyoto University

Kyoto, 606-01 Japan

Abstract We investigate the properties of $2-D$ gravity in the Batalin and Vilkovisky quantization scheme. We find a factorized structure which exhibits duality properties analogous to those existing in the topological theories of forms. New conformal field are introduced with their invariant action.

*email address: baulieu@lpthe.jussieu.fr

† Boite 126, Tour 16, 1^{er} étage, 4 place Jussieu, F-75252 Paris CEDEX 05, FRANCE

1 Introduction

In a previous paper, we have shown that the Batalin and Vilkovisky (B-V) formalism [1] for the gauge theories of forms coupled to Yang-Mills fields can be formulated in a unifying algebraic framework where the fields and the anti-fields are assembled into dual pairs [2]. We have obtained a powerful algorithm which generates topological actions of the Chern-Simon and Donaldson-Witten type on the basis of vanishing curvature conditions.

In this paper we extend our analysis to conformal theories coupled to 2-D gravity. We show in particular that the ghost system and the Wess and Zumino action of 2-D gravity possess a structure similar to that of a Chern-Simon action and we point out the possibility of introducing new conformal fields specific to 2-D gravity, with an invariant action. This action has a new gauge symmetry which complements the ordinary conformal invariance. Our work uses the Beltrami parametrization which gives a quasi Yang-Mills structure to the gauge invariances of conformal theories and preserves the factorization properties between the holomorphic and anti-holomorphic sectors.

The paper is organized as follows. We first briefly review the description of conformal 2-D gravity with Wess and Zumino terms in the framework of the Beltrami parametrization. Then we incorporate these results in the Batalin and Vilkovisky quantization scheme and find a duality picture between fields and anti-fields similar to the one we had found in [2] for the theories of forms. All relevant conformal fields and anti-fields, including the new fields suggested by our unification procedure, build up dual pairs and can be assembled as the components of differential forms with a grading equal to the sum of the ghost number and of the ordinary form degree.

2 2-D gravity in the Beltrami parametrization

$2-D$ gravity has some very interesting algebraic properties when the metric is expressed in terms of the Beltrami differential [3]. Let us briefly recall the known results. The Beltrami parametrization of the metrics in 2-dimensional space means that one expresses the length of

line elements on the worldsheet as

$$ds^2 = \Lambda(dz + \mu_{\bar{z}}^{z} d\bar{z})(d\bar{z} + \mu_{z}^{\bar{z}} dz) \tag{2.1}$$

$\Lambda(z)$ is the conformal factor, $\mu_{\bar{z}}^{z}$ is the Beltrami differential and z denotes the complex coordinates.

This parametrization has the following advantages. It permits one to build a theory which includes the Wess and Zumino field related to the conformal anomaly and never refers to the conformal factor $\Lambda(z)$. Moreover, one can define a tensor calculus which only refers to the Beltrami parametrization [3] [4].

The BRST transformation laws of the Beltrami differential $\mu_{\bar{z}}^{z}(z)$ and of its ghost $c^z(z)$ take the following factorized form

$$\begin{aligned} s\mu_{\bar{z}}^{z} &= \partial_{\bar{z}} c^z + c^z \partial_z \mu_{\bar{z}}^{z} - \mu_{\bar{z}}^{z} \partial_z c^z \\ s c^z &= c^z \partial_z c^z \end{aligned} \tag{2.2}$$

The relation between the Beltrami ghosts $c^z(z)$ and $c^{\bar{z}}(z)$ and the $2D$-reparametrization vector field ghost ξ^α is [3]

$$\begin{aligned} c^z &= (\exp i_\xi - 1)(dz + \mu_{\bar{z}}^{z} d\bar{z}) = \xi^z + \mu_{\bar{z}}^{z} \xi^{\bar{z}} \\ c^{\bar{z}} &= (\exp i_\xi - 1)(d\bar{z} + \mu_{z}^{\bar{z}} dz) = \xi^{\bar{z}} + \mu_{z}^{\bar{z}} \xi^{z} \end{aligned} \tag{2.3}$$

This parametrization of the conformally invariant part of the metric and of the ghosts is very useful to gauge-fix the conformally invariant part of the metric since it preserves automatically the factorization properties, and only depends on conformally invariant variables. Moreover, if one chooses a gauge where the Beltrami differential is equal to a given background in the holomorphic sector, the corresponding ghost action is simply

$$s(b_{zz}\mu_{\bar{z}}^{z}) = b_{zz}(\partial_{\bar{z}} c^z + c^z \partial_z \mu_{\bar{z}}^{z} - \mu_{\bar{z}}^{z} \partial_z c^z) \tag{2.4}$$

In this gauge, the antighost b_{zz} is a quadratic differential with BRST transformation $sb_{zz} = 0$. One has similar expressions in the anti-holomorphic direction, obtained by changing $\mu_{\bar{z}}^{z}$ and c^z into $\mu_{z}^{\bar{z}}$ and $c^{\bar{z}}$. The conformal gauge is recovered for $\mu_{\bar{z}}^{z} = 0$ and $\mu_{\bar{z}}^{z}$ is the source of the energy-momentum tensor components T_{zz}. In the remaining of this paper we will only consider the equations of the holomorphic sector. The equations of the anti-holomorphic sector would

follow simply by complex conjugation. These results which are specific to 2-D reparametrization invariance have a natural explanation in the context of differential geometry [3]. Note that the gauge fixing of global zero modes can be done along these lines, as explained in [5]

Let us see now that the symmetry equations can be written in a compact way, by unifying forms and ghosts. One defines

$$\begin{aligned} \hat{\mu}^z &= dz + \mu_{\bar{z}}^z d\bar{z} + c^z \\ \tilde{d} &= d + s \end{aligned} \tag{2.5}$$

The curvature of $\hat{\mu}^z$ is

$$\hat{F}^z = (d+s)\hat{\mu}^z - \frac{1}{2}\{\hat{\mu}^z, \hat{\mu}^z\} = \tilde{d}\hat{\mu}^z - \hat{\mu}^z \partial_z \hat{\mu}_z \tag{2.6}$$

The BRST transformation laws defined in eq. (2.2) are just the vanishing curvature condition

$$\hat{F}^z = 0 \tag{2.7}$$

One should observe that $F^z = F_{\bar{z}} dz d\bar{z}$ vanishes identically, with no restriction on $\mu_{\bar{z}}^z$.

The existence and the expression of the conformal anomaly under a factorized form derive from the following descent equations

$$\hat{I}_4 = (s+d)\hat{\Delta}_3 = 0 \tag{2.8}$$

with

$$\hat{\Delta}_3 = \hat{\Gamma}_z^z \tilde{d}\hat{\Gamma}_z^z = \hat{\mu}^z \partial_z \hat{\mu}^z \partial_z^z \hat{\mu}^z \tag{2.9}$$

and

$$\hat{\Gamma}_z^z = \partial_z \hat{\mu}_z^z \tag{2.10}$$

One has indeed $\tilde{d}\hat{\Gamma}_z^z = \hat{\mu}^z {\partial_z}^2 \hat{\mu}^z$ which implies that $\tilde{d}\tilde{\Gamma}_z^z \tilde{d}\hat{\Gamma}_z^z$ vanishes identically because $\hat{\mu}^z \hat{\mu}^z = 0$. The consistent left conformal anomaly is thus the two-form component with ghost number one of $\hat{\Delta}_3$. It can be written as [3]

$$\Delta_2^1 = dz d\bar{z} (\partial_z c^z {\partial_z}^2 \mu_{\bar{z}}^z - \partial_z \mu_{\bar{z}}^z {\partial_z}^2 c^z) \tag{2.11}$$

To obtain a conformal Lagrangian whose BRST variation reproduces the anomaly, one defines a Wess and Zumino scalar field L with the following one form field-strength

$$\hat{G} = \tilde{d}L - \hat{\mu}\partial L - a\partial_z \hat{\mu}^z = \tilde{d}L - \{\hat{\mu}, L\} - a\hat{\Gamma}_z^z \tag{2.12}$$

a is any given real number. This curvature satisfies the Bianchi identity

$$\tilde{d}\hat{G} = \hat{\mu}^z \partial_z \hat{G} \tag{2.13}$$

The classical field-strength $G = G_{\overline{z}} d\overline{z}$ and the BRST transformation of the field L are defined by

$$\hat{G} = G_{\overline{z}} d\overline{z} \tag{2.14}$$

Indeed, the ghost decomposition of this equation is

$$G_{\overline{z}} = (\partial_{\overline{z}} - \mu_{\overline{z}}^{z} \partial_z) L - a \partial_z \mu_{\overline{z}}^{z} \tag{2.15}$$

and

$$sL = c^z \partial_z L - a \partial_z c^z. \tag{2.16}$$

The last equation can be understood as the definition of the Wess and Zumino field L.

The transformation law of the field strength $G = G_{\overline{z}} d\overline{z}$ can be deduced directly from the Bianchi identity satisfied by $\hat{G}$

$$sG_{\overline{z}} = c^z \partial_z G_{\overline{z}} \tag{2.17}$$

The possibility of using L as a Wess and Zumino field follows from the equation

$$a^2 \hat{\Delta}_3 = \tilde{d}(\hat{\mu}^z \partial_z L(\hat{G} + a \partial_z \hat{\mu}^z) - a \hat{G} \partial_z \hat{\mu}^z) \tag{2.18}$$

This equation can be obtained by inserting the relation $a \hat{\Gamma}_z^z = \tilde{d}L - \hat{\mu}\partial L - \hat{G}$ in the expression of $\hat{\Delta}_3$. By expansion in ghost number and form degree, one gets

$$\begin{aligned} a^2 \hat{\Delta}_3 \quad = \quad & \tilde{d} \; [\; (\partial_z L (\partial_{\overline{z}} - \mu_{\overline{z}}^{z} \partial_z) L - 2a \partial_z \mu_{\overline{z}}^{z}) \; dz d\overline{z} \\ & + (c^z \partial_z L ((\partial_{\overline{z}} - \mu_{\overline{z}}^{z} \partial_z) L - 2a \partial_z \mu_{\overline{z}}^{z}) \\ & - a \partial_z c^z ((\partial_{\overline{z}} - \mu_{\overline{z}}^{z} \partial_z) L - a \partial_z \mu_{\overline{z}}^{z}) + \mu_{\overline{z}}^{z} \partial_z L) \; d\overline{z} \\ & - a \partial_z L \partial_z c^z \; dz - a \partial_z L \; c^z \partial_z c^z \;] \end{aligned} \tag{2.19}$$

The introduction of the field L has therefore the consequence of rendering trivial, i.e, equal to a sum of s-exact and d-exact terms, all components obtained from the ghost expansion of the closed three-form $\hat{\Delta}_3$.

One has in particular

$$a^2 \int \Delta^1_2 = a^2 \int dz d\overline{z} \, \left[\partial_z c^z \partial_z^2 \mu_{\overline{z}}^z - \partial_z \mu_{\overline{z}}^z \partial_z^2 c^z \right]$$
$$= \int dz d\overline{z} \; s \left[\partial_z L \left((\partial_{\overline{z}} - \mu_{\overline{z}}^z \partial_z) \, L - 2a \partial_z \mu_{\overline{z}}^z \right] \qquad (2.20)$$

Therefore

$$\mathcal{L}_{WZ} = \partial_z L \left((\partial_{\overline{z}} - \mu_{\overline{z}}^z \partial_z) L - 2a \partial_z \mu_{\overline{z}}^z \right)$$
$$= \partial_z L G_{\overline{z}} - a \partial_z \mu_{\overline{z}}^z \partial_z L \qquad (2.21)$$

can be thought as a Wess-Zumino Lagrangian density which can counterbalance the conformal anomaly.

Now, it is easy to verify that $\int dz d\overline{z} \; e^{-\frac{L}{a}}$ is an invariant action, owing to the transformation law of L defined in eq. (2.16). Such a term is analogous to the cosmological term of the Liouville action. It does not contribute to the classical energy momentum tensor, since it is independent of $\mu_{\overline{z}}^z$.

If, furthermore, we introduce a conformal field H_z with

$$s H_z = \partial_z (c^z H_z) \qquad (2.22)$$

we find that

$$\int dz d\overline{z} \; H_z G_{\overline{z}} \qquad (2.23)$$

is an invariant action.

Putting everything together, we are led to consider the following action

$$\int dz d\overline{z} \; \Big[\quad (H_z + \beta \partial_z L) \left((\partial_{\overline{z}} - \mu_{\overline{z}}^z \partial_z) L - a \partial_z \mu_{\overline{z}}^z \right) + \alpha e^{-\frac{L}{a}}$$
$$-\beta a \partial_z \mu_{\overline{z}}^z \partial_z L$$
$$-b_{zz} \left(\partial_{\overline{z}} c^z + c^z \partial_z \mu_{\overline{z}}^z - \mu_{\overline{z}}^z \partial_z c^z \right) \Big] \qquad (2.24)$$

The first term ensures the propagation of L and H_z and the second one is a cosmological type interaction. The last term is the ordinary $b-c$ ghost system resulting from the gauge fixing of the Beltrami differential. The terms which would remain for $\beta = 0$ are BRST invariant. The contributions of all terms proportional to β are such that their BRST variations reproduce the

conformal anomaly. Both coefficients a and α can be set equal to one by field redefinitions and one has the freedom to chose at will the value of the parameter β to compensate for a conformal anomaly, resulting for instance from a coupling to other fields.

The energy-momentum tensor is obtained by differentiating this action with respect to $\mu_{\bar z}^{z}$

$$\begin{aligned} T_{zz} &= -(H_z+\beta\partial_z L)\partial_z L + a\partial_z H_z + 2a\beta\partial_z^2 L \\ &\quad +\partial_z(b_{zz}c^z) + b_{zz}\partial_z c^z \end{aligned} \tag{2.25}$$

Once again, it should be stressed that no reference to the conformal factor of the metrics is necessary in this approach, and that mirror equations exist in the anti-holomorphic sector.

3 B-V approach to 2-D gravity in the Beltrami parametrization

We will show now that the above action can be naturally interpreted in the B-V formalism. We will closely follow the ideas introduced in [2] where fields and anti-fields appear in dual combinations, in a way which is consistent with the unification of fields into forms, with a grading equal to the sum of the ghost number (which is a negative integer for the anti-fields) and of the ordinary form degree. In this section we will consider a minimal set of fields which reproduces the results of the previous section. In the next section we will see that new fields and their invariant action can be introduced if one pushes further our principle of unification.

Let us consider the Beltrami differential and ghost generalized one-form that we have already defined in the previous section

$$\hat\mu^z = dz + \mu_{\bar z}^{z} d\bar z + c^z \tag{3.1}$$

According to [2], it is natural to combine the anti-fields M_{zz}^{-1} and $M_{zz\bar z}^{-2}$ of $\mu_{\bar z}^{z}$ and c^z into the following generalized zero-form, "dual" to $\hat\mu^z$

$$\hat M_z = M_{zz}^{-1} dz + M_{zz\bar z}^{-2} dz d\bar z \tag{3.2}$$

One must also introduce the anti-fields $H_{z\bar z}^{-1}$ and $L_{\bar z}^{-1}$ of the Wess and Zumino sector fields L and H_z. This leads us to introduce the following generalized zero-form and one-form, "dual" to each other

$$\begin{aligned} \hat L &= L_{\bar z}^{-1} d\bar z + L \\ \hat H &= H_z dz + H_{z\bar z}^{-1} dz d\bar z. \end{aligned} \tag{3.3}$$

In the next section, we will show that other field components can occur in the expansion of the forms in eqs. (3.1), (3.2) and (3.3). In this sense, the pairs of fields and anti-fields $(\mu_{\bar{z}}^{z}, M_{zz}^{-1})$ and $(c^z, M_{zz\bar{z}}^{-2})$ in the pure gravity sector, and $(L_{\bar{z}}^{-1}, H_z)$ and $(L, H_{z\bar{z}}^{-1})$ in the Wess and Zumino sector, build up a minimal system.

The curvatures of these generalized forms are

$$
\begin{aligned}
\mathcal{F}^z &= (s+d)\hat{\mu}^z - \frac{1}{2}\{\hat{\mu}^z, \hat{\mu}^z\} = (s+d)\hat{\mu}^z - \hat{\mu}^z\partial_z\hat{\mu}^z \\
\mathcal{D}\hat{M}_z &= (s+d)\hat{M}_z - \{\hat{\mu}^z, \hat{M}_z\} - \{\hat{H}, L\} \\
&= (s+d)\hat{M}_z - \mu^z\partial_z\hat{M}_z + 2\hat{M}_z\partial_z\mu^z - \hat{H}\partial_z\hat{L} \\
\mathcal{D}\hat{L} &= (s+d)\hat{L} - \{\hat{\mu}^z, \hat{L}\} = (s+d)\hat{L} - \hat{\mu}^z\partial_z\hat{L} \\
\mathcal{D}\hat{H} &= (s+d)\hat{H} - \{\hat{\mu}^z, \hat{H}\} = (s+d)\hat{H} - \partial_z(\hat{\mu}^z\hat{M})
\end{aligned}
\tag{3.4}
$$

These definitions are consistent with Bianchi identities.

The BRST symmetry is then defined by the following constraints on the curvatures

$$
\begin{aligned}
\mathcal{F}^z &= 0 \\
\mathcal{D}\hat{M}_z &= a\partial_z\hat{H} \\
\mathcal{D}\hat{L} &= a\partial_z\hat{\mu}^z \\
\mathcal{D}\hat{H} &= 0
\end{aligned}
\tag{3.5}
$$

The explicit form of the action of s on all fields and anti-fields is obtained by expanding eqs. (3.5) in form degree and ghost number. The property $s^2 = 0$ is the consequence the Jacobi relation satisfied by the graded bracket $\{\ ,\ \}$ appearing in eq. (3.4), with relations of the type

$$
\mathcal{F}^z = \mathcal{D}\mathcal{D} = 0 \tag{3.6}
$$

and $\mathcal{D}(\partial_z\tilde{\mu}^z) = \partial_z(\tilde{d}\hat{\mu}) - \{\hat{\mu}, \partial_z\hat{\mu}\} = 0$. One recovers of course the the same transformation laws of the fields as in section (2).

This construction of the BRST symmetry is justified by its efficiency and also by the fact that it follows the same pattern as for many other types of gauge symmetries.

Let us denote generically all fields and ghosts by ϕ and their anti-fields by ϕ^*. We will show

that the BRST operation defined in eq. (3.5) is associated to the following B-V action

$$S[\phi,\phi^*]=\int\Big[\hat{M}_z(d\hat{\mu}^z+\frac{1}{2}\{\hat{\mu},\hat{\mu}\}^z)+\hat{H}(d\hat{L}-\hat{\mu}\partial\hat{L}-a\partial_z\hat{\mu}^z)\Big]_2^0 \tag{3.7}$$

Indeed, if one defines

$$F^{z(\hat{\mu})} \;=\; d\hat{\mu}^z-\frac{1}{2}\{\hat{\mu}^z,\hat{\mu}^z\} \tag{3.8}$$

and

$$G^{\hat{L}}=d\hat{L}-\hat{\mu}\partial\hat{L}-a\partial_z\hat{\mu}^z \tag{3.9}$$

one can rewrite the action (3.7) as

$$S[\phi,\phi^*] \;=\; \int\Big[\hat{M}_z\;F^{z(\hat{\mu})}+\hat{H}G^{\hat{L}}\Big]_2^0 \tag{3.10}$$

Then, by using the definition of s given by eq. (3.5) one can verify

$$\int s\;\Big(\hat{M}_z\;F^{z(\hat{\mu})}+\hat{H}G^{\hat{L}}\Big)=0 \tag{3.11}$$

The component with ghost number one of this equation gives the wanted result that the B-V action (3.7) is BRST invariant.

Reciprocally, the action (3.7) contains the information about the BRST symmetry, through the B-V equations

$$s\phi=\frac{\delta S[\phi,\phi^*]}{\delta\phi^*}\qquad s\phi^*=\frac{\delta S[\phi,\phi^*]}{\delta\phi} \tag{3.12}$$

Let us verify this. After expansion in ghost number of all fields, the action (3.10) is

$$\begin{aligned}S[\phi,\phi^*]=\int dzd\bar{z}\quad[\quad & H_z\left((\partial_{\bar{z}}-\mu_{\bar{z}}^z\partial_z)L-a\partial_z\mu_{\bar{z}}^z\right)\\ &+M_{zz}^{-1}(\partial_{\bar{z}}c^z+c^z\partial_z\mu_{\bar{z}}^z-\mu_z^z\partial_z c^z)\\ &+M_{zz\bar{z}}^{-2}c^z\partial_z c^z\\ &+H_{z\bar{z}}^{-1}(c^z\partial_z L-a\partial_z c^z)\\ &+L_{\bar{z}}^{-1}\partial_z(c^zH_z)\quad]\end{aligned} \tag{3.13}$$

The action (3.13) has a linear dependance in the anti-fields which means that it is of the first rank in the B-V sense. The BRST transformations of all fields with positive ghost number (the

quantum field theory propagating fields) are thus the field polynomials which appear in factor of the anti-fields in eq. (3.13). It is then immediate to verify that if one applies the relations (3.12) to the action (3.13), one gets the same expression for the action of s on all the fields and anti-fields as the one obtained from eqs. (3.5).

The B-V procedure allows one to add a term proportional to $e^{-\frac{L}{a}}$ to $S[\phi, \phi^*]$ since such a term is compatible with the symmetry. This would modify the constraint on the curvature of $\tilde{L}$ in eq. (3.5) by the addition of the equation of motion stemming from this invariant term. It is also allowed to add to the B-V action the Wess and Zumino term defined in eq. (2.21) if one wishes to produce a theory which can compensate a conformal anomaly.

The B-V formalism indicates how one can introduce antighosts to perform the gauge-fixing. Since the only ghost is c^z, we have only one possible gauge function and thus only one possible antighost in the holomorphic sector. The freedom in the choice of the gauge function allows one to introduce the antighost as a quadratic differential b_{zz} in view of reaching for example the gauge where $\mu_{\bar{z}}^{z}$ is set equal to a background value $\mu_{\bar{z}0}^{z}$. A term $b_{zz}^{*}\lambda_{\bar{z}}^{z}$ should be added to the action, where $\lambda_{\bar{z}}^{z}$ is a Nakanishi-Lautrup type Lagrange multiplier field and b_{zz}^{*} is the anti-field of b_{zz}. The gauge fixed action will be obtained by introducing the gauge function

$$Z^{-1} = b_{zz}(\mu_{\bar{z}}^{z} - \mu_{\bar{z}0}^{z}) \tag{3.14}$$

and by replacing all anti-fields by mean of the constraint $\phi^* = \delta Z^{-1}/\delta\phi$. As a result, the anti-field M_{zz}^{-1} is set equal to the usual antighost b_{zz} and all other anti-fields are zero. One eventually recovers the action defined in eq. (2.24).

The s-transformation of the anti-field M_{zz}^{-1} is

$$sM_{zz}^{-1} = \frac{\delta \mathcal{L}_{z\bar{z}}}{\delta \mu_{\bar{z}}^{z}} = T_{zz} \tag{3.15}$$

From our point of view, this equation explains the fact that the energy-momentum tensor T_{zz} is a Q-commutator in the Hamiltonian formalism [6].

Apart from technical details, the interesting result of this section is the simplicity of the B-V action defined in eq. (3.10). This action, which contains the whole information about the transformation laws of the field of 2-D gravity including the Wess and Zumino sector, is analogous to a Chern-Simon action. This is not too much a surprise in view of earlier results, where the relevance of the equation $\hat{\mu}\tilde{d}\hat{\mu} = 0$ had been emphasized for building the BRST

algebra of conformal models [3]. This relationship is probably related to the topological nature of the ghost and Wess and Zumino sector sector of string theory.

Let us conclude this section by indicating how our formulae permit a straightforward derivation of $2-D$ topological gravity equations [7]. Following ref. [2], one introduces the topological ghosts of 2-D gravity as the components with positive ghost number of a 2-form $\hat{X}_2^z = \Psi^z_{\bar{z}} d\bar{z} + \Phi^z$. The associated anti-fields are then the components of a dual (-1)-form $\hat{Y}_{-1z} = \Psi^{*-2}_{zz} dz + \Phi^{*-3}_{\bar{z}zz} dz d\bar{z}$. The invariant B-V action is then

$$\int \Big[\hat{M}_z (F^{z\hat{\mu}} + \hat{X}_2^z) + \hat{X}_2^z D^{\hat{\mu}} \hat{Y}_{-1z}) \Big]_2^0 \tag{3.16}$$

The transformation laws for the fields and anti-fields which leave invariant this action are defined by the curvature constraints

$$\begin{aligned}
\hat{\mathcal{F}}^z &= \tilde{d}\hat{\mu} - \frac{1}{2}\{\hat{\mu}, \hat{\mu}\} = \hat{X}_2^z \\
\mathcal{D}\hat{M}_z &= \tilde{d}\hat{M}_z - \{\hat{\mu}, \hat{M}_z\} = \{\hat{X}_2^z, \hat{Y}_{-1z}\} \\
\mathcal{D}\hat{X}_2^z &= \tilde{d}\hat{X}_2^z - \{\hat{\mu}, \hat{X}_2^z\} = 0 \\
\mathcal{D}\hat{Y}_{-1z} &= \tilde{d}\hat{Y}_{-1z} - \{\hat{\mu}, \hat{Y}_{-1z}\} = 0
\end{aligned} \tag{3.17}$$

These formulae give the BRST transformation laws of the topological 2-D gravity in the Beltrami parametrization as in ref. [8]. With suitable choices of gauge functions, the gauge fixing of the B-V action ([**?**]) would reproduce the known actions for 2-D topological gravity.

4 A more general action for 2-D gravity with Wess and Zumino terms

We have just found how to incorporate in a rather simple algebraic framework all fields and anti-fields relevant to 2-D gravity, including the Wess and Zumino sector. Our basic tools have been the Beltrami parametrization of the conformally invariant part of the metric and the unification of fields and anti-fields into forms graded by the sum of their ordinary form degree and ghost number. The latter quantity is positive for the ordinary ghosts and negative for their anti-fields, which explains why some components of the forms have higher ordinary form degree.

However, by looking at eqs. (3.1, 3.2, 3.3), we see that we have restricted our-selves in the expansion of forms since $\hat{\mu}^z$, $\hat{M}_z$, $\hat{L}$ and $\hat{H}_z$, which are respectively generalized 1-form, 0-form, 0-form and 1-form, could contain additional components matching the grading requirements. When we have defined the BRST symmetry by imposing constraints on the curvatures of these forms, these restrictions have yield no contradiction because of the identity $d\mu^z - \mu^z\partial_z\mu^z = 0$. It is thus quite natural to generalize the field contents by considering instead of eqs. (3.1, 3.2, 3.3) the following general form decomposition

$$
\begin{aligned}
\tilde{\mu}^z &= \mu_{\bar{z}}^{-1} dz d\bar{z} + \nu dz + (dz + \mu_{\bar{z}}^z d\bar{z}) + c^z \\
\tilde{M}_z &= M_z + M_{z\bar{z}}^{-1} d\bar{z} + M_{zz}^{-1} dz + M_{zz\bar{z}}^{-2} dz d\bar{z} \\
\tilde{L} &= L + L_{\bar{z}}^{-1} d\bar{z} + L_z^{-1} dz + L_{z\bar{z}}^{-2} dz d\bar{z} \\
\tilde{H} &= H^1 + H_{\bar{z}} d\bar{z} + H_z dz + H_{z\bar{z}}^{-1} dz d\bar{z}
\end{aligned}
\tag{4.1}
$$

Let us look at the components with positive ghost number in these expansions. There are three new classical fields with ghost number zero, namely ν which has conformal weight zero, and M_z and H_z which have conformal weights one. Then, there is H^1 which has ghost number one, and which represents an additional gauge freedom.

Thus, besides the pairs of fields and anti-fields of the previous section $(\mu_{\bar{z}}^z, M_{zz}^{-1})$, $(c^z, M_{zz\bar{z}}^{-2})$, $(H_z, L_{\bar{z}}^{-1})$ and $(L, H_{z\bar{z}}^{-1})$, we have now the pairs $(\nu, M_{z\bar{z}}^{-1})$, $(M_z, \mu_{\bar{z}}^{-1})$, $(H_{\bar{z}}, L_z^{-1})$ and $(H^1, L_{z\bar{z}}^{-2})$. The introduction of the object νdz in the expansion of the Beltrami differential will imply a non vanishing value for classical component of the Beltrami curvature.

The action of the BRST symmetry on all the fields is given by the same curvature constraints as in the previous section. The property $(s+d)^2 = 0$ still holds, since the new fields have been introduced just as new components of the differential forms (4.1) and the constraints are compatible with the Bianchi identities. One has therefore

$$
\begin{aligned}
\mathcal{F}^z &= (s+d)\tilde{\mu}^z - \frac{1}{2}\{\tilde{\mu}^z, \tilde{\mu}^z\} \\
&= (s+d)\tilde{\mu}^z - \tilde{\mu}^z\partial_z\tilde{\mu}^z = 0 \\
\mathcal{D}\tilde{M}_z &= (s+d)\tilde{M}_z - \{\tilde{\mu}^z, \tilde{M}_z\} - \{\tilde{H}, \tilde{L}\} \\
&= (s+d)\tilde{M}_z - \mu^z\partial_z\tilde{M}_z + 2\tilde{M}_z\partial_z\tilde{\mu}^z - \tilde{H}\partial_z\tilde{L} = a\partial_z\tilde{H} \\
\mathcal{D}\tilde{L} &= (s+d)\tilde{L} - \{\tilde{\mu}^z, \tilde{L}\}
\end{aligned}
$$

$$
\begin{aligned}
&= (s+d)\tilde{L} - \tilde{\mu}^z\partial_z\tilde{L} = a\partial_z\tilde{\mu}^z \\
\mathcal{D}\tilde{H} &= (s+d)\tilde{H} - \{\tilde{\mu}^z, \tilde{H}\} \\
&= (s+d)\tilde{H} - \partial_z(\tilde{\mu}^z\tilde{H}) = 0
\end{aligned} \tag{4.2}
$$

The corresponding B-V action is the same as in eq. (3.10), except that the forms have now a more general decomposition in the fields and anti-fields. Using the decomposition given by eq. (4.1), one gets the following B-V action

$$
\begin{aligned}
S[\phi,\phi^*] = \int dz d\overline{z} \quad [\quad & M_z\,((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)\nu + \nu\partial_z\mu_{\overline{z}}^z) \\
&+H_z((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)L - a\partial_z\mu_{\overline{z}}^z) \\
&+H_{\overline{z}}(a\partial_z\nu + \nu\partial_z L) \\
&+M_{z\overline{z}}^{-1}(\partial_{\overline{z}}c^z + c^z\partial_z\mu_{\overline{z}}^z - \mu_{\overline{z}}^z\partial_z c^z) \\
&+M_{z\overline{z}}^{-1}(c^z\partial_z\nu - \nu\partial_z c^z) \\
&+\mu_{\overline{z}}^{-1}(c^z\partial_z M_z + 2M_z\partial_z c^z - H^1\partial_z L - a\partial_z H^1) \\
&+M_{zz\overline{z}}^{-2}c^z\partial_z c^z \\
&+H_{z\overline{z}}^{-1}(c^z\partial_z L - a\partial_z c^z) \\
&+L_{\overline{z}}^{-1}(\partial_z(c^z H_z - \nu H^1) \\
&+L_z^{-1}\left((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)H^1 - H^1\partial_z\mu_{\overline{z}}^z + \partial_z(c^z H_{\overline{z}})\right) \\
&+L_{z\overline{z}}^{-2}\partial_z(c^z H^1)\ ]
\end{aligned} \tag{4.3}
$$

Let us consider the classical part of this action

$$
\begin{aligned}
S[\phi,\phi^*=0] = \int dz d\overline{z} \quad [\quad & M_z((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)\nu + \nu\partial_z\mu_{\overline{z}}^z) \\
&+H_z((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)L - a\partial_z\mu_{\overline{z}}^z) \\
&+H_{\overline{z}}(a\partial_z\nu + \nu\partial_z L)\ \ ]
\end{aligned} \tag{4.4}
$$

It is invariant under the following symmetry

$$
\begin{aligned}
s\mu_{\overline{z}}^z &= \partial_{\overline{z}}c^z + c^z\partial_z\mu_{\overline{z}}^z - \mu_{\overline{z}}^z\partial_z c^z \\
s\nu &= c^z\partial_z\nu - \nu\partial_z c^z
\end{aligned}
$$

$$
\begin{aligned}
sM_z &= c^z\partial_z M_z + 2M_z\partial_z c^z - H^1\partial_z L - a\partial_z H^1 \\
sH_z &= \partial_z(c^z H_z - \nu H^1) \\
sH_{\overline{z}} &= (\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)H^1 - H^1\partial_z\mu_{\overline{z}}^z + \partial_z(c^z H_{\overline{z}}) \\
sL &= c^z\partial_z L - a\partial_z c^z
\end{aligned}
\tag{4.5}
$$

We see that in addition to the reparametrization invariance governed by the ghost c^z we have another gauge symmetry governed by the ghost H^1, with $sH^1 = \partial_z(c^z H^1)$ and that $H_{\overline{z}}$ plays the role of a gauge field for this symmetry. It is quite interesting that the action can be written as

$$
S[\phi, \phi^* = 0] = \int dz d\overline{z}\, [M_z F_{\overline{z}} + H_z G_{\overline{z}} + H_{\overline{z}} G_z]
\tag{4.6}
$$

with

$$
\begin{aligned}
F_{\overline{z}} &= (\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)\nu + \nu\partial_z\mu_{\overline{z}}^z \\
G_{\overline{z}} &= (\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)L - a\partial_z\mu_{\overline{z}}^z \\
G_z &= \nu\partial_{\overline{z}} L + a\partial_z L
\end{aligned}
\tag{4.7}
$$

The invariance of the action can be verified from

$$
\begin{aligned}
sF_{\overline{z}} &= c^z\partial_z F_{\overline{z}} - F_{\overline{z}}\partial_z c^z \\
sG_{\overline{z}} &= c^z\partial_z G_{\overline{z}} \\
sG_z &= c^z\partial_z G_z
\end{aligned}
\tag{4.8}
$$

and there are non trivial compensations between the variations of the three terms of the action which involve the ghost H^1.

For a further clarification of this formula, let us notice that we can summarize in the following compact way the transformation laws of $\mu_{\overline{z}}^z$ and ν

$$
\begin{aligned}
\tilde{F}^z &= (s+d)(\nu dz + dz + \mu_{\overline{z}}^z d\overline{z} + c^z) \\
&\quad -\frac{1}{2}\{\nu dz + dz + \mu_{\overline{z}}^z d\overline{z} + c^z, \nu dz + dz + \mu_{\overline{z}}^z d\overline{z} + c^z\} \\
&\quad = F^z = F_{\overline{z}} dz d\overline{z}
\end{aligned}
\tag{4.9}
$$

and

$$(s+d)\tilde{F}^z = \{\nu dz + dz + \mu_{\overline{z}}^z d\overline{z} + c^z, \tilde{F}^z\} \tag{4.10}$$

The relation $sF_{\overline{z}} = c^z\partial_z F_{\overline{z}} - F_{\overline{z}}\partial_z c^z$ is the component with ghost number one of eq. (4.10). There are similar formulae which involve G_z and $G_{\overline{z}}$.

A very direct way to prove the invariance of the action is to define

$$F^{z\tilde{\mu}} \quad = \quad d\tilde{\mu}^z - \frac{1}{2}\{\tilde{\mu}^z, \tilde{\mu}^z\} \tag{4.11}$$

and

$$G^{\tilde{L}} = d\tilde{L} - \tilde{\mu}\partial\tilde{L} - a\partial_z\tilde{\mu}^z \tag{4.12}$$

and to check

$$\int s\left[\tilde{M}_z\ F^{z(\tilde{\mu})} + \tilde{H}G^{\tilde{L}}\right] = 0 \tag{4.13}$$

The proof is quite straightforward from the relations

$$dF^{z\tilde{\mu}} = \{\tilde{\mu}^z, F^{z\tilde{\mu}}\} \tag{4.14}$$

$$dG^{\tilde{L}} = \{\tilde{\mu}^z, G^{\tilde{L}}\} + \{L, F^{z\tilde{\mu}}\} - a\partial_z F^{z\tilde{\mu}} \tag{4.15}$$

The introduction of the field ν implies therefore that one has a non vanishing classical Beltrami curvature $F^{\overline{z}} = ((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)\nu + \nu\partial_z\mu_{\overline{z}}^z)\, dz d\overline{z}$, contrarily to what happens in the usual case which involves a restricted number of fields.

Another distinction is the existence of a new degree of gauge freedom associated to the ghost H^1. This freedom can be used for instance to gauge-fix to zero the field $H_{\overline{z}}$. To do so one introduces an antighost $\overline{r}_z$ associated to H^1, its anti-field $r_{\overline{z}}^*$ and the associated Lagrange multiplier field β_z, with $s\overline{r}_z = \beta_z$. Then one adds to the B-V action the term $r_{\overline{z}}^*\beta_z$. The gauge fixed action is finally obtained by considering the gauge function

$$Z^{-1} = b_{zz}(\mu_{\overline{z}}^z - \mu_{\overline{z}0}^z) + \overline{r}_z H_{\overline{z}} \tag{4.16}$$

By replacing all anti-fields in the B-V action (4.3) by $\phi^* = \delta Z^{-1}/\delta\phi$, one obtains the following action

$$\int dz d\overline{z} \quad [\quad M_z\, ((\partial_{\overline{z}} - \mu_{\overline{z}}^z\partial_z)\nu + \nu\partial_z\mu_{\overline{z}}^z)$$

$$+H_z\left((\partial_{\bar z}-\mu_{\bar z}^z\partial_z)L-a\partial_z\mu_{\bar z}^z\right)$$

$$-b_{zz}(\partial_{\bar z}c^z+c^z\partial_z\mu_{\bar z}^z-\mu_{\bar z}^z\partial_z c^z)$$

$$-\overline{r}_z\Big((\partial_{\bar z}-\mu_{\bar z}^z\partial_z)H^1-H^1\partial_z\mu_{\bar z}^z+\partial_z(c^zH_z)\Big)\ \Big] \tag{4.17}$$

This action is of the conformal type. In addition to the usual propagating pairs $b_{zz}-c^z$ and H_z-L we have now the pairs $M_z-\nu$ and $\overline{r}_z-H^1$. The last term in the action is the ghost term corresponding to the gauge-fixing to zero of $H_{\bar z}$. Other types of gauge-fixing for the field $H_{\bar z}$ could be defined which would lead us to different ghost interactions for H^1.

The form of the consistent anomaly $\tilde\Delta_3$, with $(d+s)\tilde\Delta_3=0$ is unchanged, since the basic structure equations are the same, and we have not found an anomaly in the symmetry parametrized by the ghost H^1. The same Wess and Zumino Lagrangian density as in the restricted theory is thus applicable to this model to make it anomaly free.

Our construction has therefore led us to propose the following action for the 2-D gravity, which includes a Wess and Zumino term and a cosmological type term

$$\begin{aligned}S_{2D}=\int dz d\bar z\quad\Big[\ & (H_z+\beta\partial_z L)((\partial_{\bar z}-\mu_{\bar z}^z\partial_z)L-a\partial_z\mu_{\bar z}^z)\\ & -\beta a\partial_z\mu_{\bar z}^z\partial_z L+\alpha e^{-\frac{L}{a}}\\ & +M_z((\partial_{\bar z}-\mu_{\bar z}^z\partial_z)\nu+\nu\partial_z\mu_{\bar z}^z)\\ & -b_{zz}(\partial_{\bar z}c^z+c^z\partial_z\mu_{\bar z}^z-\mu_{\bar z}^z\partial_z c^z)\\ & -\overline{r}_z\Big((\partial_{\bar z}-\mu_{\bar z}^z\partial_z)H^1-H^1\partial_z\mu_{\bar z}^z+\partial_z(c^zH_z)\Big)\ \Big]\end{aligned} \tag{4.18}$$

The associated energy momentum tensor is

$$\begin{aligned}T_{zz}\ =\ & -(H_z+\beta\partial_z L)\partial_z L+a\partial_z H_z+2a\beta\partial_z^2 L\\ & -M_z\partial_z\nu-\partial_z(M_z\nu)\\ & +\partial_z(b_{zz}c^z)+b_{zz}\partial_z c^z\\ & +\partial_z(\overline{r}_zH^1)\end{aligned} \tag{4.19}$$

These expressions for the Lagrangian and energy momentum tensor should be complemented by their mirror expressions, obtained by complex conjugation. a and α can be set equal to one as in the case with the restricted set of fields, while β can be choosen at will, possibly with different values in both holomorphic and anti-holomorphic sectors.

5 conclusion

We have applied the B-V formalism for 2-D gravity with a Wess and Zumino sector. By using the Beltrami parametrization of conformal field theories, we have found the same type of unification between all fields and anti-fields as the one we had previously observed in [2] for the theories of forms coupled to Yang-Mills fields. Moreover, we have shown that the B-V action has a structure quite similar to that of a Chern-Simon action. We have introduced new conformal fields with a conceptually very simple action. In addition to the ordinary conformal invariance, this action has a new gauge symmetry and induces new ghost interactions, with a possible assymmetry between the holomorphic and anti-holomorphic sectors. Its properties and its possible couplings to matter will be studied in a separate publication.

Acknowledgments: The author would like to express his deep gratitude to RIMS for the hospitality extended to him during his stay in Japan.

References

[1] I.A. Batalin and V.A. Vilkovisky, *Phys. Rev.* **D28** (1983) 2567.

[2] L. Baulieu, PAR–LPTHE 95–55 preprint, Field Anti-Field Duality and Topological Field Theories.

[3] L. Baulieu and M. Bellon, *Phys. Lett.* **B196** (1987) 142. L. Baulieu, M. Bellon and R. Grimm *Phys.Lett.* **B228** (1989) 325; R. Stora in "Proceedings of the 1987 Cargese summer school on non perturbative field theory", eds. G. t'Hooft et al., Nato ASI Series B, Phys. Vol. 185, p.433; L. Baulieu, same proceedings.

[4] H. Nicolai *Nuclear Physics* **B414** (1994) 299; R. Dick *Lett. Math. Phys.* **18** (1989) 67, *Fortschr. Physik* **40** (1992) 519.

[5] L. Baulieu and M. Bellon, *Phys. Lett.* **B202** (1988) 67.

[6] R. Dijkgraaf, E. Verlinde and H. Verlinde *Nuclear Phys.* **B352** (1991) 59.

[7] J. Labastida, M. Pernici, and E. Witten, *Nucl. Phys.* **B310**, 611, (1988).

[8] L. Baulieu and I.M. Singer, *Comm. of Math. Phys.* **135**, 253 (1991).